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Explicit solution for the potential flow due to an assembly of stirrers in an inviscid fluid

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Abstract The motion of an irrotational incompressible fluid driven by an assembly of stirrers, of arbitrary shape, moving at specified velocities in the fluid is considered. The problem is shown to be equivalent to a standard mathematical problem in potential theory known as the modified Schwarz problem. It turns out that the solution to this problem can be written down, in closed form, as an explicit integral depending on the conformal mapping to the fluid region from a canonical pre-image region and a kernel function expressed in terms of a transcendental function called the Schottky–Klein prime function. In this way, an explicit integral solution, up to conformal mapping, for the complex potential of the flow generated by an arbitrary assembly of stirrers can be written down.

Keywords Ideal flow · Multiply connected · Schwarz problem · Stirrers

1 Introduction

There has been a recent revival of interest in computing the irrotational flow generated by objects moving in a planar incompressible inviscid fluid. Wang [1] recently considered the problem of the fluid motion generated by two moving cylindrical objects. Burton et al. [2] studied the same problem from a mathematical-physics viewpoint. An independent resurgence of interest in this basic fluid-dynamical problem has also occurred since Boyland et al. [3] proposed the use of a circular disk-containing ideal fluid and a finite number of circular disk "stirrers" as a basic model (a "batch stirring device") for understanding the topological fluid mechanics associated with mixing. Other studies on such systems have been performed subsequently (e.g. [4]).

The purpose of this paper is to show that there exists an explicit integral formula, up to conformal mapping, for the solution to this problem for an arbitrary collection of M stirrers in any bounded domain (or indeed for any collection of stirrers in an unbounded domain). This is done by showing that this particular fluid-dynamics problem is equivalent to a celebrated general problem in potential theory known as the *modified Schwarz problem*. An integral formula for the solution to this problem has recently been reported by the author [5] using a special transcendental function known as the *Schottky–Klein prime function* (in fact, this same function will be used, in a different context, in the present paper). The form of this explicit solution is not only compact but also easily

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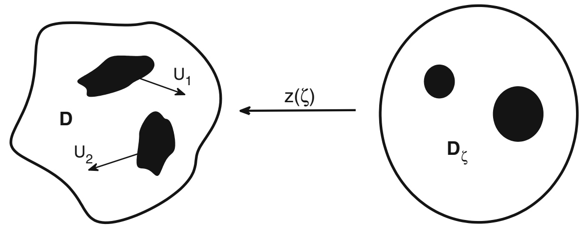

Fig. 1 A general fluid region D with two moving stirrers of arbitrary shape and the pre-image circular domain D_ζ which maps to D under mapping $z(\zeta)$. D_ζ is a triply connected circular domain

implementable in practice. The efficacy of this formula for practical computation of the particular class of flow problems considered here is demonstrated by a series of examples in Sect. 6.

In the case of just two stirrers (so that the fluid region is doubly connected) considered by Wang [1] and Burton et al. [2], the problem admits a celebrated integral-formula solution known as the *Villat formula* [6]—a fact recently pointed out by Crowdy et al. [7]. The Villat formula is only relevant to doubly connected domains. This paper exploits a generalization of the Villat formula to higher-connected domains [5] to produce explicit integral formulae for the irrotational flow generated by *more* than two stirrers (in fact, any finite number of stirrers).

The key result of this paper is to give an integral representation for the complex potential $w(z)$ associated with M stirrers moving in an inviscid fluid in a given planar region D in an (x, y)-plane. Here $z = x + \mathrm{i}y$. Any such region is known to be conformally equivalent, under a conformal mapping $z = z(\zeta)$, to a multiply connected circular domain D_ζ in a complex parametric ζ-plane (a circular domain is one whose boundaries are all circular). Figure 1 shows a schematic for the case of two stirrers ($M = 2$). We suppose the map $z(\zeta)$ is known explicitly (or can be constructed using standard conformal mapping techniques). Then, in this ζ-plane, if the j-th stirrer is moving with complex velocity (u_j, v_j), so that its complex velocity is $U_j = u_j + \mathrm{i}v_j$, the complex potential $W(\zeta) \equiv w(z(\zeta))$ for the flow, as a function of ζ, is given by

$$W(\zeta) = \frac{1}{2\pi}\oint_{C_0}\left[\mathfrak{Re}[-\mathrm{i}\overline{U_0}z(\zeta')]\right]\left(\mathrm{d}\log\omega(\zeta',\zeta) + \mathrm{d}\log\overline{\omega}(\zeta'^{-1},\zeta^{-1})\right)$$
$$-\sum_{j=1}^{M}\frac{1}{2\pi}\oint_{C_j}\left[\mathfrak{Re}[-\mathrm{i}\overline{U_j}z(\zeta')] + d_j\right]\left(\mathrm{d}\log\omega(\zeta',\zeta) + \mathrm{d}\log\overline{\omega}(\overline{\theta_j}(\zeta'^{-1}),\zeta^{-1})\right), \tag{1}$$

where the constants $\{d_j | j = 1, \ldots, M\}$ solve an M-by-M linear system given explicitly in Sect. 5, $\omega(.,.)$ is a special function introduced in Sect. 4 (it is known as the Schottky–Klein prime function [8]) and $\{C_j | j = 0, 1, \ldots, M\}$ are the circular boundaries of D_ζ. (Formula (1) also allows for the outer boundary C_0 to be moving at some complex speed U_0). (1) is an explicit integral representation of the complex potential $W(\zeta)$. It is then straightforward, if $z(\zeta)$ is known, to find values of the complex potential $w(z)$ (hence the fluid velocity) in the original (x, y)-plane.

Actually, there are many other problems in fluid dynamics where the modified Schwarz problem (or its dual, the regular Schwarz problem) arises and the results derived here can be applied there too. For example, Crowdy and Marshall ([9,10]) have presented a novel analytical approach to computing N-vortex motion in arbitrary multiply connected domains bounded by islands or impenetrable walls. This is a Hamiltonian dynamical system and the Hamiltonians (or *Kirchhoff–Routh path functions*) can be written down in analytical form in terms of the Schottky–Klein prime function [9]. The effects of background flows can be incorporated into the governing Hamiltonians additively. If some application involves the vortices evolving in a non-stationary domain—where the boundary walls are moving according to some external influence, for example—then the results of this paper can be used to find analytical expressions for these additional contributions to the governing Hamiltonian.

2 The problem of fluid stirrers

We seek an incompressible, irrotational flow in the fluid region D containing M solid objects $\{D_j | j = 1, \ldots, M\}$ which are supposed to be moving with externally specified velocities. There exists a velocity potential ϕ such that

$\mathbf{u} = \nabla\phi$ in D. Let ∂D_j denote the boundary of the object D_j while the outer boundary of D will be denoted ∂D_0. The flow must satisfy the boundary conditions

$$\mathbf{u.n} = \mathbf{U_j.n}, \quad \text{on } \partial D_j \tag{2}$$

where, for $j = 0, 1, \ldots, M$, $\mathbf{U_j} = (u_j, v_j)$ is the externally specified velocity of object D_j. Since the flow is incompressible, $\nabla^2\phi = 0$ in D. In this problem the fluid region is multiply connected so there is also the freedom to specify the circulations around the objects $\{D_j | j = 1, \ldots, M\}$. For now, we take these circulations to be zero. The generalization to non-zero circulations around the stirrers is elucidated in Sect. 3.

Suppose $\mathbf{u} = (u, v)$ and consider a complex-variable formulation of the problem. In what follows, if $\mathbf{a} = (a_x, a_y)$ denotes a vector then the notation a will be used to denote its natural complex analogue $a = a_x + \mathrm{i}a_y$. It is well-known that the solution of the mathematical problem just stated is equivalent to finding a complex potential $w(z)$, analytic in D, satisfying the boundary conditions (2) where $\mathrm{d}w/\mathrm{d}z = u - \mathrm{i}v$. Since we have additionally imposed that there is no circulation around any of the objects $\{D_j | j = 1, \ldots, M\}$, it follows that $w(z)$ will not only be analytic but also single-valued in D. The complex form of the unit tangent vector is z_s where s denotes arclength (taken in an appropriate direction around the boundary contour), while the complex form of the unit normal is $-\mathrm{i}z_s$. Note that, if $\mathbf{a}$ and $\mathbf{b}$ denote two vectors (with complex counterparts a and b) then $\mathbf{a.b} = \mathfrak{Re}[\overline{a}b]$. It follows that conditions (2) become

$$\mathfrak{Re}\left[\frac{\mathrm{d}w(z)}{\mathrm{d}z}(-\mathrm{i}z_s)\right] = \mathfrak{Re}[\overline{U_j}(-\mathrm{i}z_s)], \quad \text{on } \partial D_j, \quad j = 0, 1, \ldots, M. \tag{3}$$

An important observation is that these boundary conditions can be integrated with respect to the arclength s to give

$$\mathfrak{Re}\,[-\mathrm{i}w(z)] = \mathfrak{Re}[-\mathrm{i}\overline{U_j}z] + d_j, \quad \text{on } \partial D_j, \quad j = 0, 1, \ldots, M, \tag{4}$$

where $\{d_j | j = 0, 1, \ldots, M\}$ are real constants of integration (more generally, in an unsteady flow, these will be functions of time, but not space). Without loss of generality, we can set just one of these constants, d_0 say, to zero.

It is a consequence of the Riemann mapping theorem that any multiply connected domain D in a complex z-plane can be mapped to, via some conformal map $z(\zeta)$, from a bounded pre-image circular domain D_ζ in the ζ-plane [11]. Let this preimage domain D_ζ comprise the unit ζ-disk with M smaller circular disks excised. Let C_0 denote the unit circle $|\zeta| = 1$ and let $\{C_j | j = 1, \ldots, M\}$ denote the boundaries of the M enclosed circular disks. The centres and radii of the interior circular disk will be denoted $\{\delta_j | j = 1, \ldots, M\}$ and $\{q_j | j = 1, \ldots, M\}$, respectively. A schematic of such a pre-image region D_ζ is shown in Fig. 2. We will suppose that C_0 maps to the boundary ∂D_0 with all other boundaries $\{\partial D_j | j = 1, \ldots, M\}$ being the images of the other circles $\{C_j | j = 1, \ldots, M\}$. Actually, our results will also apply to any *unbounded* fluid region containing a total of $M + 1$ stirrers. In this case, the only difference is that the conformal map $z(\zeta)$ will contain a simple pole singularity inside D_ζ which will map to the point at infinity in the fluid region, while C_0 will map to the boundary of an additional stirrer.

Let us introduce

$$W(\zeta) \equiv w(z(\zeta)). \tag{5}$$

Then (4) becomes the following boundary-value problem for the function $W(\zeta)$ that is single-valued and analytic in D_ζ:

$$\begin{aligned} &\mathfrak{Re}\,[-\mathrm{i}W(\zeta)] = \mathfrak{Re}[-\mathrm{i}\overline{U_0}z(\zeta)], \quad \text{on } C_0 \\ &\mathfrak{Re}\,[-\mathrm{i}W(\zeta)] = \mathfrak{Re}[-\mathrm{i}\overline{U_j}z(\zeta)] + d_j, \quad \text{on } C_j, \quad j = 1, \ldots, M. \end{aligned} \tag{6}$$

This should be recognized as precisely the *modified Schwarz problem* [13] of classical potential theory for a single-valued analytic function in D_ζ given its real part on the boundary ∂D_ζ, up to a set of constants. The solution for a single-valued analytic function $W(\zeta)$ exists provided the constants $\{d_j | j = 1, \ldots, M\}$ satisfy certain compatibility conditions.

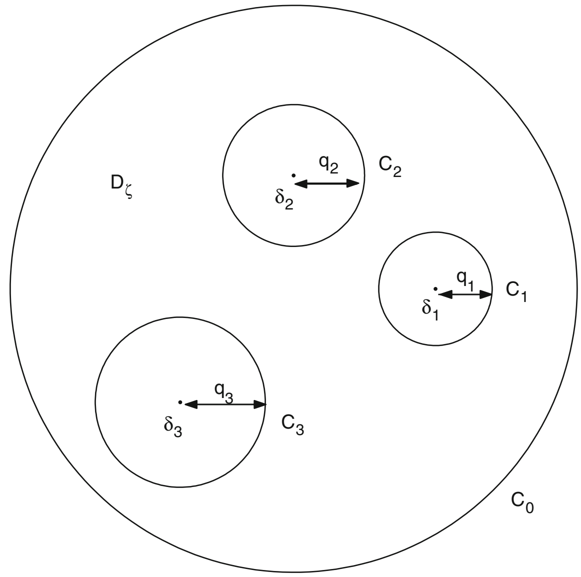

Fig. 2 A typical multiply connected circular domain D_ζ. The case shown is quadruply connected

3 The modified Schwarz problem

The usual statement of the modified Schwarz problem [12,13] is to find a single-valued analytic function $f_s(\zeta)$ in some domain given information about its real part on the boundary of the domain, specifically,

$$\begin{aligned} \mathfrak{Re}\,[f_s(\zeta)] &= \phi_0, \quad \text{on } C_0, \\ \mathfrak{Re}\,[f_s(\zeta)] &= \phi_j + d_j, \quad \text{on } C_j, \quad j = 1, \ldots, M, \end{aligned} \tag{7}$$

where $\{\phi_j | j = 0, 1, \ldots, M\}$ are some given real-valued functions and $\{d_j | j = 1, \ldots, M\}$ is a set of constants that must be chosen in a special way in order that a solution for a single-valued $f_s(\zeta)$ exists. The compatibility conditions determining the set $\{d_j | j = 1, \ldots, M\}$ are given explicitly in Sect. 5.

Crowdy [5] has shown that an explicit integral solution of the modified Schwarz problem can be written down in terms of the Schottky–Klein prime function associated with the circular domain D_ζ. This function is defined in the next section. For the purposes of this paper we will simply write down, and make use of, this integral solution. It can be written concisely in the form

$$\begin{aligned} f_s(\zeta) = &\frac{1}{2\pi \mathrm{i}} \oint_{C_0} \phi_0 \left(\mathrm{d}\log \omega(\zeta', \zeta) + \mathrm{d}\log \overline{\omega}(\zeta'^{-1}, \zeta^{-1}) \right) \\ &- \sum_{j=1}^{M} \frac{1}{2\pi \mathrm{i}} \oint_{C_j} (\phi_j + d_j) \left(\mathrm{d}\log \omega(\zeta', \zeta) + \mathrm{d}\log \overline{\omega}(\overline{\theta_j}(\zeta'^{-1}), \zeta^{-1}) \right) + \mathrm{i}C, \end{aligned} \tag{8}$$

where C is some real constant and $\omega(\zeta, .)$ is the aforementioned Schottky–Klein prime function associated with the circular domain D_ζ. For completeness, a derivation of (8) is recorded in Appendix A (however, readers only interested in computing the flow field generated by an assembly of stirrers can simply proceed in knowledge of the fact that (8) is the required solution).

It follows from the general formula (8) that an expression for the complex potential $W(\zeta)$ relevant to the problem of fluid stirrers considered here is, to within an unimportant constant, given by (1).

For completeness, we now indicate how to generalize our approach to the case where there happens to be a non-zero circulation γ_k around D_k. Within this model, these circulations can be arbitrarily satisfied (however, to be consistent with Kelvin's circulation theorem, these circulations should be constant in time). Let the complex potential $W(\zeta)$ now be decomposed as follows:

$$W(\zeta) = \sum_{k=1}^{M} -\frac{\mathrm{i}\gamma_k}{2\pi}\log(\zeta - \delta_k) + \mathcal{W}(\zeta), \tag{9}$$

where $\{\gamma_k | k = 1, \ldots, M, \gamma_k \in \mathbb{R}\}$ are the required circulations around the stirrers and where $\mathcal{W}(\zeta)$ is a single-valued analytic function in D_ζ. Note that $W(\zeta)$ itself is no longer single-valued in D_ζ. It is easy to verify that such a decomposition yields the correct circulation around the stirrers. To see this, note that

$$\oint_{\partial D_k} \mathbf{u}.\mathbf{dl} = \oint_{\partial D_k} \mathfrak{Re}\left[\frac{\mathrm{d}w}{\mathrm{d}z}\mathrm{d}z\right] = \mathfrak{Re}\oint_{\partial D_k} \mathrm{d}w = \mathfrak{Re}\oint_{C_k} \mathrm{d}W = \gamma_k, \tag{10}$$

where we have used (9) and the residue theorem to obtain the final equality. Now, the boundary-value problem for $W(\zeta)$ can be rewritten as a boundary-value problem for the single-valued analytic function $\mathcal{W}(\zeta)$, i.e.,

$$\begin{aligned} \mathfrak{Re}\,[-\mathrm{i}\mathcal{W}(\zeta)] &= \mathfrak{Re}[-\mathrm{i}\overline{U_0}z(\zeta)] - \sum_{k=1}^{M}\frac{\gamma_k}{2\pi}\log|\zeta - \delta_k|, \quad \text{on } C_0, \\ \mathfrak{Re}\,[-\mathrm{i}\mathcal{W}(\zeta)] &= \mathfrak{Re}[-\mathrm{i}\overline{U_j}z(\zeta)] - \sum_{k=1}^{M}\frac{\gamma_k}{2\pi}\log|\zeta - \delta_k| + d_j, \quad \text{on } C_j,\ j = 1, \ldots, M. \end{aligned} \tag{11}$$

But this is just a modified Schwarz problem for the single-valued analytic function $\mathcal{W}(\zeta)$ (very much akin to (6)) for which an integral formula analogous to (1) can be written down. Thus, non-zero circulations around the stirrers provide no additional difficulty.

4 The Schottky–Klein prime function

We now introduce the Schottky–Klein prime function. First, define M Möbius maps $\{\phi_j | j = 1, \ldots, M\}$ corresponding to the conjugation map for points on the circle C_j. That is, if C_j has equation

$$|\zeta - \delta_j|^2 = (\zeta - \delta_j)(\bar{\zeta} - \bar{\delta}_j) = q_j^2, \tag{12}$$

then

$$\bar{\zeta} = \bar{\delta}_j + \frac{q_j^2}{\zeta - \delta_j} \tag{13}$$

and so

$$\phi_j(\zeta) \equiv \bar{\delta}_j + \frac{q_j^2}{\zeta - \delta_j}. \tag{14}$$

If ζ is a point on C_j, then its complex conjugate is given by $\bar{\zeta} = \phi_j(\zeta)$.
Next, introduce the Möbius maps

$$\theta_j(\zeta) \equiv \overline{\phi}_j(\zeta^{-1}) = \delta_j + \frac{q_j^2\zeta}{1 - \bar{\delta}_j\zeta}. \tag{15}$$

Let C_j' be the circle obtained by reflection of the circle C_j in the unit circle $|\zeta| = 1$ (i.e., the circle obtained by the transformation $\zeta \mapsto 1/\bar{\zeta}$). It is easily verified that the image of the circle C_j' under the transformation θ_j is the circle C_j; indeed, $\theta_j(\zeta)$ maps points on C_j' to points on C_j. Since the M circles $\{C_j | j = 1, \ldots, M\}$ are non-overlapping, so are the M circles $\{C_j' | j = 1, \ldots, M\}$. The (classical) *Schottky group* Θ is defined to be the infinite free group of mappings generated by compositions of the $2M$ basic Möbius maps $\{\theta_j | j = 1, \ldots, M\}$ and their inverses $\{\theta_j^{-1} | j = 1, \ldots, M\}$ and including the identity map.

One way to define the Schottky–Klein prime function is by a classical infinite product formula given in Baker [8] in the form

$$\omega(\zeta, \gamma) = (\zeta - \gamma)\hat{\omega}(\zeta, \gamma), \tag{16}$$

where

$$\hat{\omega}(\zeta,\gamma) \equiv \prod_{\theta_i \in \Theta''} \frac{(\theta_i(\zeta)-\gamma)(\theta_i(\gamma)-\zeta)}{(\theta_i(\zeta)-\zeta)(\theta_i(\gamma)-\gamma)} \tag{17}$$

and where the product is over all mappings θ_i in the set Θ'' which denotes all mappings in the Schottky group Θ excluding the identity and all inverse maps. This means that if $\theta_1\theta_2$ is included, say, then $\theta_2^{-1}\theta_1^{-1}$ (its inverse) must be excluded. It is necessary to truncate the product (17) in order to write a function routine to evaluate it. A natural way to truncate it is by the *level* of the mappings in the Schottky group. The identity map is the single level-zero map. The maps $\{\theta_k, \theta_k^{-1} | k = 1, \dots, M\}$ are the $2M$ level-one maps. Any composition of these level-one maps that does not reduce to the identity is a level-two map. By extension, a composition of any *three* of the level-one maps that does not reduce to a lower level map is called a level-three map, and so on. Provided the product is convergent, truncating it just at the level-3 maps can give 5 to 6 digits of accuracy in the evaluation of the prime function. This is sufficient for many applications.

While formulae (16) and (17) can be used effectively for many domains D_ζ, the infinite product (17) does not converge for all choices of D_ζ. However, the Schottky–Klein prime function is a well-defined function for all choices of domain D_ζ and Crowdy and Marshall [14] have devised a new method of computing it. The algorithm described in [14] is an alternative method for computing $\omega(\zeta, .)$ when the infinite product representation (16) and (17) is not convergent (or converges too slowly to be practicable).

5 Computation of $\{d_j | j = 1, \dots, M\}$

An important step is the evaluation of the constants $\{d_j | j = 1, \dots, M\}$. These are determined by a set of compatibility conditions on the given boundary data of the modified Schwarz problem which must hold if the solution is to exist. In Appendix A, the explicit form of these compatibility conditions on the given boundary data is derived. These M conditions are

$$\oint_{\partial D} \phi \frac{\partial \sigma_j}{\partial n} \mathrm{d}s = 0, \quad j = 1, \dots, M, \tag{18}$$

where the functions $\{\sigma_j | j = 1, \dots, M\}$ are known as the *harmonic measures* associated with D. These are defined in Appendix A.

Crowdy and Marshall [15] have shown how to express the M harmonic measures $\{\sigma_j | j = 1, \dots, M\}$ in terms of the Schottky–Klein prime function. It turns out that conditions (18) are equivalent to the conditions

$$\oint_{C_0} \phi_0 \frac{\partial \tilde{\sigma}_k}{\partial n}\, \mathrm{d}s - \sum_{j=1}^{M} \oint_{C_j} (\phi_j + d_j) \frac{\partial \tilde{\sigma}_k}{\partial n}\, \mathrm{d}s = 0, \quad k = 1, \dots, M, \tag{19}$$

where

$$\tilde{\sigma}_k(\zeta) = \log \left| \frac{\omega(\zeta, \overline{\phi_k(\alpha)})}{\omega(\zeta, \bar{\alpha}^{-1})} \right| \tag{20}$$

for some arbitrarily chosen point $\alpha \in D_\zeta$ (the functions (20) can be shown not to depend on the choice of the point α). On use of the fact that

$$\frac{\partial \tilde{\sigma}_k}{\partial n} \mathrm{d}s = \Im\mathrm{m} \left[2 \frac{\partial \tilde{\sigma}_k}{\partial \zeta} \mathrm{d}\zeta \right] = \Im\mathrm{m} \left(\frac{\omega_\zeta(\zeta, \overline{\phi_k(\alpha)})}{\omega(\zeta, \overline{\phi_k(\alpha)})} \mathrm{d}\zeta - \frac{\omega_\zeta(\zeta, \bar{\alpha}^{-1})}{\omega(\zeta, \bar{\alpha}^{-1})} \mathrm{d}\zeta \right), \tag{21}$$

conditions (19) reduce to a simple linear system with coefficients that can be determined explicitly in terms of integrals involving $\omega(\zeta, .)$. Indeed, the linear system for the column vector $\mathbf{d} \equiv (d_1, \dots, d_M)^T$ can be written

$$\mathbf{A} \cdot \mathbf{d} = \mathbf{b}, \tag{22}$$

where the matrix **A** and vector **b** have components

$$
\begin{aligned}
A_{kj} &= \mathfrak{Im}\left[\oint_{C_j}\left(\frac{\omega_\zeta(\zeta,\overline{\phi_k(\alpha)})}{\omega(\zeta,\overline{\phi_k(\alpha)})}\mathrm{d}\zeta - \frac{\omega_\zeta(\zeta,\bar{\alpha}^{-1})}{\omega(\zeta,\bar{\alpha}^{-1})}\mathrm{d}\zeta\right)\right],\\
b_k &= -\mathfrak{Im}\left[\sum_{j=1}^{M}\mathfrak{Re}[-\mathrm{i}\overline{U_j}z(\zeta)]\left(\frac{\omega_\zeta(\zeta,\overline{\phi_k(\alpha)})}{\omega(\zeta,\overline{\phi_k(\alpha)})}\mathrm{d}\zeta - \frac{\omega_\zeta(\zeta,\bar{\alpha}^{-1})}{\omega(\zeta,\bar{\alpha}^{-1})}\mathrm{d}\zeta\right)\right].
\end{aligned}
\tag{23}
$$

With the vector **d** determined in this way, formula (1) can now be used to find $W(\zeta)$.

6 Examples

To check the viability of using the integral formula (1) in practice, we recompute some streamline distributions calculated by Finn et al. [4]. The latter authors computed such potential flows in order to study the mixing properties associated with various stirring protocols in different "batch stirring devices". They studied stirring devices consisting of the unit disk containing different numbers of circular disk stirrers. In the present formulation, such domains are exactly the circular domains D_ζ (so that the conformal map for this case is just the identity map, i.e., $z(\zeta) = \zeta$). Figure 2 of [4] shows some typical streamline distributions and, while the authors did not specify the precise geometry or the velocities of the stirrers, these can be approximated by inspection of their Fig. 2. Figures 3–5 show streamline distributions for 2, 3 and 4 circular disk stirrers in the fluid contained in the unit disk. The geometrical configurations have been chosen to be as close as possible to the first three featured in Fig. 2 of Finn et al. [4]. Moreover, the same instantaneous directions of the stirrer motions have also been used (with the speed set equal to 1 in all cases). The circulations around the stirrers has been assumed to vanish. Reassuringly, it is found that the qualitative nature of the streamline distributions is very similar.

It should be pointed out that it is possible to retrieve, using the present formulation, the representation of the complex potential $W(\zeta)$ used by Finn et al. [4] in their numerical computations of the required irrotational flows. To do this, we decompose $W(\zeta)$ as

$$
W(\zeta) = \sum_{n=0}^{\infty} a_n^{(0)}\zeta^n + \sum_{j=1}^{M}\sum_{n=1}^{\infty}\frac{a_n^{(j)}q_j^n}{(\zeta-\delta_j)^n}
\tag{24}
$$

and consider a truncation of the infinite sums of the Fourier–Laurent series at order N_1 where N_1 is chosen to be sufficiently large for the desired accuracy (typically, we choose $N_1 = 20$). To determine the coefficients $\{a_n^{(0)}|n = 0, 1, \ldots, N_1\}$ and $\{a_n^{(j)}|n = 1, \ldots, N_1;\ j = 1, \ldots, M\}$ a distribution of N_2 collocation points inside, or on the boundary of, D_ζ can be picked and (24) evaluated at each of these points. The left-hand side of (24) can be calculated at these collocation points using the explicit integral formula (1). The resulting (linear) system of N_2 equations can then be solved for the required coefficients using a least-squares algorithm. We must choose $N_2 \geq (M+1)N_1 + 1$ in order to obtain an overdetermined linear system. In fact, if it is required to evaluate $W(\zeta)$ repeatedly at a large number of points in D_ζ, it is advisable to compute the coefficients in the expansion (24) as just described and then use (24) for the evaluations. This is because, once the coefficients in (24) have been determined, evaluating (24) is computationally very efficient.

Of course, the authors of [4] derive the coefficients of (24) by substituting this representation in the boundary conditions and enforcing them at a number of collocation points and then solving the resulting linear system by a least-squares method. Nevertheless, we believe it is important to recognize, as we have shown here, that the very same solution has an explicit integral representation in terms of special functions.

The method given here is not restricted just to stirrers that are circular disks, but can be applied to *any* set of stirrers, even in *unbounded* fluid domains. All that is required is knowledge of the conformal map from a circular region D_ζ to the fluid region D. To highlight this, Fig. 6 shows the instantaneous streamlines of the irrotational flow generated by three "paddles" or thin plates moving in an unbounded fluid region. The conformal map in this case

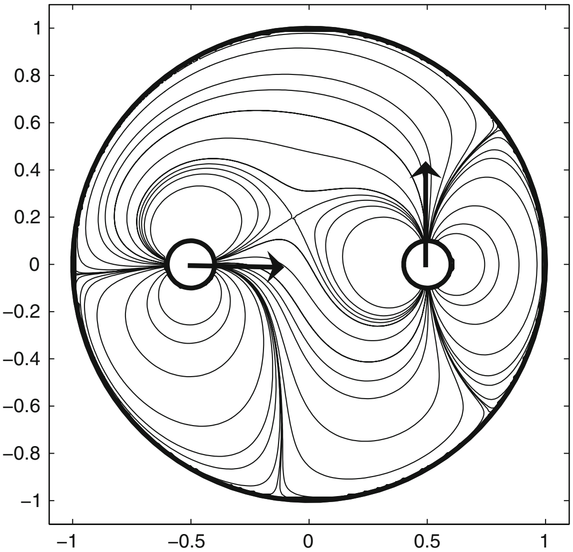

Fig. 3 Streamlines of the irrotational flow generated by two circular disk stirrers. The fluid domain is triply connected. The stirrers are at: $z_1 = -0.5$, radius 0.1, (complex) speed 1, $z_2 = 0.5$, radius 0.1, (complex) speed i. Qualitatively, the streamline distribution compares well with [4, Fig. 2]

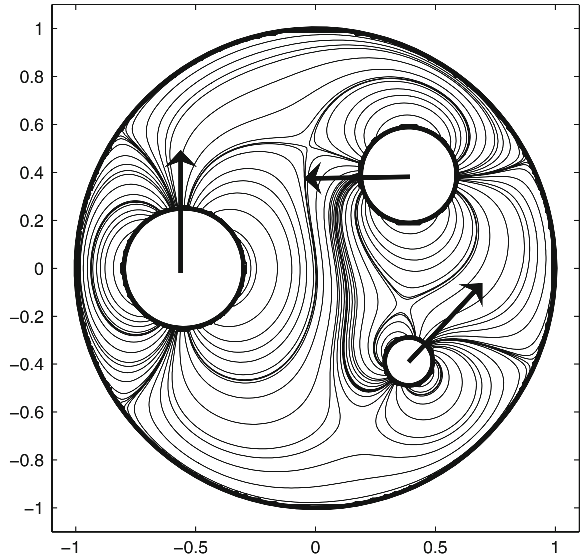

Fig. 4 Streamlines of the irrotational flow generated by three circular disk stirrers. The fluid domain is quadruply connected. The stirrers are at $z_1 = -0.55$, radius 0.25, (complex) speed i, $z_2 = 0.55\mathrm{e}^{\mathrm{i}\pi/4}$, radius 0.2, (complex) speed -1, $z_3 = 0.55\mathrm{e}^{-\pi\mathrm{i}/4}$, radius 0.1, (complex) speed $\mathrm{e}^{\mathrm{i}\pi/4}$. Qualitatively, the streamline distribution compares well with [4, Fig. 2]

is from the circular domain D_ζ consisting of the unit disk with two circular disks of radius 0.05 centred at ± 0.25 excised. The map $z(\zeta)$ is given explicitly by

$$z(\zeta) = \frac{\omega^2(\zeta, -1) + \omega^2(\zeta, 1)}{\omega^2(\zeta, -1) - \omega^2(\zeta, 1)}, \tag{25}$$

where $\omega(\zeta, .)$ is the Schottky–Klein prime function associated with D_ζ. The derivation of such a conformal map is described in [10]. This map has a simple pole inside D_ζ mapping to the point at infinity in the fluid region. In the symmetrical arrangement in Fig. 6, the two paddles at either side of the central paddle are moving vertically with equal speeds but opposite directions. The central paddle is stationary. These streamlines are plotted by substituting (25) in (1) and computing the contours of the imaginary part of $W(\zeta)$.

7 Discussion

Formula (1), with constants $\{d_j | j = 1, \ldots, M\}$ satisfying (19) (or, equivalently, (22)) are the key results of this paper. (1) gives the complex potential for the irrotational flow generated by a collection of stirrers moving at prescribed velocities in an inviscid fluid. As we have demonstrated (cf. Fig. 6), a significant feature is that it is applicable to M stirrers of *any* shape in *any* bounded domain or, alternatively, to $M + 1$ stirrers in any unbounded domain. All that is required is knowledge (either analytical or numerical) of the conformal mapping function $z(\zeta)$ from a conformally equivalent circular domain D_ζ to the fluid region of interest. Then, for a given point ζ in D_ζ the corresponding point in the physical domain is given by $z(\zeta)$ while the complex potential at that point in the physical plane is given by $W(\zeta)$ from (1). The (complex) fluid velocity $u - \mathrm{i}v$ at that point follows from $u - \mathrm{i}v = W_\zeta(\zeta)/z_\zeta(\zeta)$.

There are a variety of numerical approaches to finding the solutions to the problems considered here (see, for example, the monograph by Prosnak [16]) but it is important to know that there exists explicit integral representations

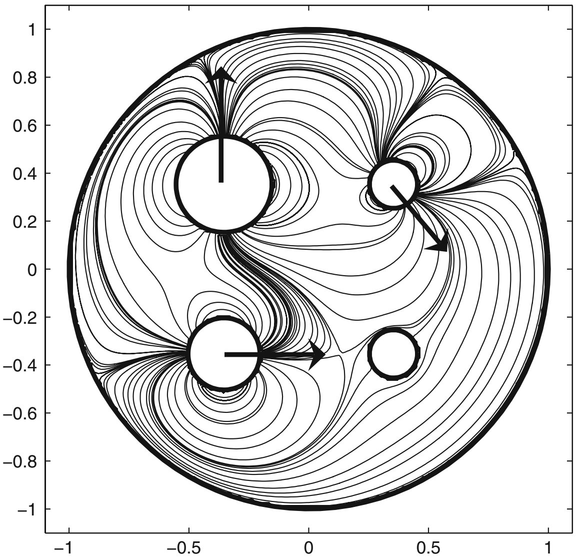

Fig. 5 Streamlines of the irrotational flow generated by four circular disk stirrers. The fluid region is quintuply connected. The stirrers are at $z_1 = 0.5\,\mathrm{e}^{3\pi \mathrm{i}/4}$, radius 0.2, (complex) speed i, $z_2 = 0.5\,\mathrm{e}^{\pi \mathrm{i}/4}$, radius 0.1, (complex) speed $\mathrm{e}^{-\mathrm{i}\pi/4}$, $z_3 = 0.5\,\mathrm{e}^{-\pi \mathrm{i}/4}$, radius 0.1, (complex) speed 0, $z_4 = 0.5\,\mathrm{e}^{-3\pi \mathrm{i}/4}$, radius 0.15, (complex) speed 1. Qualitatively, the streamline distribution compares well with [4, Fig. 2]

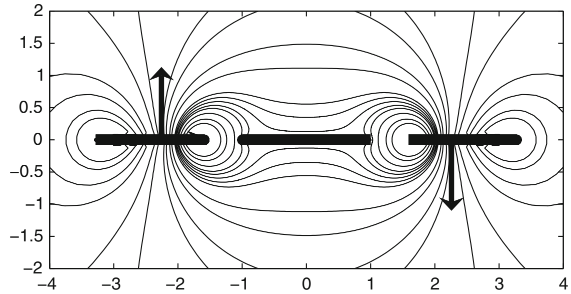

Fig. 6 Streamlines of the irrotational flow generated by three paddles in an unbounded flow. The fluid region is triply connected. The left-most paddle has (complex) speed i, the central paddle is stationary and the right-most paddle has (complex) speed $-\mathrm{i}$

for the solution, in terms of classical special functions (the Schottky–Klein prime function), once the conformal mapping $z(\zeta)$ is known.

One of the many possible applications of this work is to generalize the studies of Finn et al. [4] to the case of non-circular stirrers in their batch stirring devices (although two of the latter authors have recently presented [17] their own numerical solution to this problem, for the case of stirrers in Stokes flows, which can be adapted to the ideal flow problem considered here).

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Appendix A: Derivation of (8)

In this appendix, we give a summary of the derivation of the solution (8) of the modified Schwarz problem.

A.1 Conditions on the given data

In order for the solution to a given modified Schwarz problem to exist the given data on the domain boundaries must satisfy certain conditions which we now derive.

Suppose $\phi = \mathfrak{Re}[w(z)]$ for some analytic function $w(z)$ in D then, if ϕ is single-valued in D, so is $\partial\phi/\partial z$. So

$$\frac{\partial \phi}{\partial z} = \frac{1}{2}\frac{\mathrm{d}w}{\mathrm{d}z} \tag{26}$$

and it follows that $\mathrm{d}w/\mathrm{d}z$ is analytic and single-valued in D. Cauchy's integral formula then implies that, for any point z in D,

$$w'(z) = \frac{1}{2\pi\mathrm{i}} \oint_{\partial D} \frac{w'(\zeta)\mathrm{d}\zeta}{\zeta - z}. \tag{27}$$

By separating this integral around the boundary into a sum of the integrals around the separate boundary components, and using the geometric series expansion of $1/(\zeta - z)$ in each case, it follows from (27) that $w'(z)$ has a generalized Fourier–Laurent expansion of the form

$$w'(z) = \sum_{n\geq 0} a_n^{(0)} z^n + \sum_{k=1}^{M} \sum_{n\geq 1} \frac{a_n^{(k)}}{(z-c_k)^n}, \tag{28}$$

where c_k denotes the centre of the circle C_k and $\{a_n^{(k)}\}$ are some coefficients. But, on integration with respect to z, it is clear that the primitive $w(z)$ will itself be single-valued if and only if

$$a_1^{(k)} = 0 \quad \text{for } k = 1, \ldots, M. \tag{29}$$

These are equivalent to the conditions

$$\oint_{C_k} w'(z)\,\mathrm{d}z = \oint_{C_k} \frac{\mathrm{d}w}{\mathrm{d}s}\mathrm{d}s = \oint_{C_k} \left(\frac{\partial\phi}{\partial s} + \mathrm{i}\frac{\partial\psi}{\partial s}\right)\mathrm{d}s = 0, \quad k = 1, \ldots, M, \tag{30}$$

where ψ is the harmonic conjugate of ϕ and s denotes arclength around the boundary. Since ϕ is single-valued, it follows that ϕ must satisfy the conditions

$$\oint_{C_k} \frac{\partial\psi}{\partial s}\mathrm{d}s = \oint_{C_k} \frac{\partial\phi}{\partial n}\mathrm{d}s = 0, \quad k = 1, \ldots, M, \tag{31}$$

where $\partial/\partial n$ denotes a normal derivative to the boundary and we have made use of the Cauchy–Riemann relations.

Conditions (31) can be written in an equivalent form that depends on the boundary data for ϕ on ∂D (rather than the boundary values of $\partial\phi/\partial n$). To see this, we introduce the M *harmonic measures* $\sigma_j(z,\bar{z})$ (for $j = 1, \ldots, M$) [12] associated with D. These are defined as functions which are harmonic in D with

$$\sigma_j(z,\bar{z}) = 0 \text{ on } C_0, \quad j = 1, \ldots, M, \tag{32}$$

while for $j = 1, \ldots, M$,

$$\sigma_j(z,\bar{z}) = \begin{cases} 1 & \text{on } C_j \\ 0 & \text{on } C_k, \quad k \neq j. \end{cases} \tag{33}$$

Green's identity implies that, for each $j = 1, \ldots, M$,

$$0 = \int\int_D \left(\phi\nabla^2\sigma_j - \sigma_j\nabla^2\phi\right)\mathrm{d}V = \oint_{\partial D} \left(\phi\frac{\partial\sigma_j}{\partial n} - \sigma_j\frac{\partial\phi}{\partial n}\right)\mathrm{d}s, \tag{34}$$

where we have used the facts that ϕ and $\{\sigma_j(z,\bar{z}) | j = 1, \ldots, M\}$ are harmonic in D. However, this means that

$$\oint_{\partial D} \phi\frac{\partial\sigma_j}{\partial n}\mathrm{d}s = 0, \quad j = 1, \ldots, M, \tag{35}$$

where we have used (31), (32) and (33). The M conditions (35) must be satisfied if a solution to the modified Schwarz problem is to exist.

A.2 The modified Green's function

Let $G(z;\alpha)$ be the modified Green's function of the domain D defined to be the unique function satisfying

$$\nabla^2 G(z;\alpha) = \delta(z-\alpha), \tag{36}$$

where α is a point in D with

$$\begin{aligned} G &= 0, \quad \text{on } C_0, \\ G &= \gamma_k(\alpha), \quad \text{on } C_k, \quad k = 1, \ldots, M, \end{aligned} \tag{37}$$

where $\{\gamma_k(\alpha) | k = 1, \ldots, M\}$ are independent of z (but which generally depend on α) and are determined by the conditions

$$\oint_{C_k} \frac{\partial G}{\partial n} \mathrm{d}s = 0, \quad k = 1, \ldots, M. \tag{38}$$

Crowdy and Marshall [9] have found an explicit formula for the modified Green's function in a multiply connected circular domain D in terms of the Schottky–Klein prime function. In [9] it was applied to the problem of point vortex motion in multiply connected domains. The formula is

$$G(z; \alpha) = \mathfrak{Im}[\tilde{G}(z; \alpha)], \tag{39}$$

where

$$\tilde{G}(z; \alpha) = \frac{\mathrm{i}}{2\pi} \log \left(\frac{\omega(z, \alpha)}{|\alpha| \omega(z, \bar{\alpha}^{-1})} \right). \tag{40}$$

It follows from Green's identity that

$$\int\int_D \left(\phi \nabla^2 G - G \nabla^2 \phi \right) \mathrm{d}A = \oint_{\partial D} \left(\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n} \right) \mathrm{d}s. \tag{41}$$

On use of the properties of G, the harmonicity of ϕ in D and conditions (35), we obtain

$$\begin{aligned} \phi(\alpha, \bar{\alpha}) &= \oint_{\partial D} \left(\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n} \right) \mathrm{d}s \\ &= \oint_{\partial D} \phi \frac{\partial G}{\partial n} \mathrm{d}s - \sum_{k=1}^{M} \gamma_k \oint_{C_k} \frac{\partial \phi}{\partial n} \mathrm{d}s = \oint_{\partial D} \phi \frac{\partial G}{\partial n} \mathrm{d}s. \end{aligned} \tag{42}$$

It is a simple matter to establish that

$$\frac{\partial G}{\partial n} \mathrm{d}s = \mathfrak{Im} \left[2 \frac{\partial G}{\partial z} \mathrm{d}z \right]. \tag{43}$$

But (43), (39) and (40) imply that

$$\frac{\partial G}{\partial n} \mathrm{d}s = \mathfrak{Im} \left[-\mathrm{i} \mathrm{d}\tilde{G}(z; \alpha) \right] = \mathfrak{Re} \left[-\mathrm{d}\tilde{G}(z; \alpha) \right]. \tag{44}$$

Therefore, on substitution of (40) and (44) in (42),

$$\begin{aligned} \phi(\alpha, \bar{\alpha}) &= \mathfrak{Re} \left[\frac{1}{2\pi \mathrm{i}} \oint_{\partial D} \phi \left(\mathrm{d} \log \omega(z, \alpha) - \mathrm{d} \log \omega(z, \bar{\alpha}^{-1}) \right) \right] \\ &= \mathfrak{Re} \left[\frac{1}{2\pi \mathrm{i}} \oint_{\partial D} \phi \left(\mathrm{d} \log \omega(z, \alpha) + \mathrm{d} \log \overline{\omega}(\bar{z}, \alpha^{-1}) \right) \right], \end{aligned} \tag{45}$$

where we have simply exploited the fact that the real part of a complex number is the real part of its complex conjugate. It follows that

$$\phi(\alpha, \bar{\alpha}) = \mathfrak{Re}[f_s(\alpha)], \tag{46}$$

where

$$f_s(\alpha) = \frac{1}{2\pi \mathrm{i}} \oint_{\partial D} \phi \left(\mathrm{d} \log \omega(z, \alpha) + \mathrm{d} \log \overline{\omega}(\bar{z}, \alpha^{-1}) \right). \tag{47}$$

Alternatively, taking into account the different forms of $\bar{z}$ as a function of z on each boundary, this can be rewritten as in (8).

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